Beta-hypergeometric probability distribution on symmetric matrices

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Running title: *Beta-hypergeometric distribution*

Abstract : Some remarkable properties of the beta distribution are based on relations involving independence between beta random variables such that a parameter of one among them is the sum of the parameters of an other (see (1.1) et (1.2) below). Asci, Letac and Piccioni [1] have used the real beta-hypergeometric distribution on $\mathbb{R}$ to give a general version of these properties without the condition on the parameters. In the present paper, we extend the properties of the real beta to the beta distribution on symmetric matrices, we use on the positive definite matrices the division algorithm defined by the Cholesky decomposition to define a matrix-variate beta-hypergeometric distribution, and we extend to this distribution the proprieties established in the real case by Asci, Letac and Piccioni.

Keywords: Hypergeometric function, Beta-hypergeometric distribution, symmetric matrices, generalized power, spherical Fourier transform.

1 Introduction

Consider the gamma distribution on $\mathbb{R}$, with scale parameter $\sigma > 0$ and shape parameter $p > 0$,

$$\gamma_{p,\sigma}(dy) = \frac{\sigma^p}{\Gamma(p)} e^{-\sigma y} y^{p-1} \mathbf{1}_{(0,+\infty)}(y) dy.$$

Let U and V be two independent random variables with respective gamma distributions $\gamma_{p,\sigma}$, $\gamma_{q,\sigma}$, and define

$$X = \frac{U}{U+V} \text{ and } Y = \frac{U}{V}.$$

Then the distribution of X and Y are called the beta distributions of the first and of the second kind with parameters (p, q) and are denoted by $\beta^{(1)}_{p,q}$ and $\beta^{(2)}_{p,q}$ respectively.

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The beta distributions of the first and of second kind on $\mathbb{R}$ have many remarkable properties. For instance, it is well known (see [1]) that

$$\text{if } W' \sim \beta^{(2)}_{a+a',a'} \text{ is independent of } X \sim \beta^{(1)}_{a,a'}, \text{ then } \frac{1}{1+W'X} \sim \beta^{(1)}_{a',a}. \tag{1.1}$$

And it is shown in [3] that

$$\text{if } W \sim \beta^{(2)}_{a+a',a},\ X \sim \beta^{(1)}_{a,a'},\ W' \sim \beta^{(2)}_{a+a',a'} \text{ are independent, then } \frac{1}{1+\frac{W}{1+W'X}} \sim X. \tag{1.2}$$

In these two properties, the random variables W and W' are beta distributed with first parameter equal to the sum of the parameters of the distribution of the variable X. Asci, Letac and Piccioni [1] have used the so-called real beta-hypergeometric distribution to extend these results to the general case where $W \sim \beta^{(2)}_{b,a}$, $W' \sim \beta^{(2)}_{b,a'}$ with $b > 0$ not necessarily equal to $a + a'$. Recall that the hypergeometric function ${}_pF_q$ is defined for positive numbers $a_1, ..., a_p; b_1, ..., b_q$, by

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{n!(b_1)_n...(b_q)_n} x^n, \text{with } (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The beta-hypergeometric distribution with parameters (a, a', b) is then defined by

$$\mu_{a,a',b}(dx) = C(a, a', b)x^{a-1}(1-x)^{b-1}\ {}_2F_1(a, b; a+a'; x)\mathbf{1}_{(0,1)}(x)dx,$$

where

$$C(a, a', b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b){}_3F_2(a, a, b; a+b, a+a'; 1)}.$$

Note that the distribution $\mu_{a,a',b}$, reduces to a $\beta^{(1)}_{a,a'}$ when $b = a + a'$.
Asci, Letac and Piccioni have shown that

$$\text{if } X \sim \mu_{a,a',b} \text{ and } W' \sim \beta^{(2)}_{b,a'} \text{ are independent, then } \frac{1}{1+W'X} \sim \mu_{a',a,b} \tag{1.3}$$

and

$$\text{if } W \sim \beta^{(2)}_{b,a},\ W' \sim \beta^{(2)}_{b,a'} \text{ and } X > 0 \text{ are independent, then} \tag{1.4}$$

$$X \sim \frac{1}{1+\frac{W}{1+W'X}} \text{ if and only if } X \sim \mu_{a,a',b}.$$

In the present work, we first extend the properties in (1.1) and (1.2) to the beta distributions on symmetric matrices. We then use results from harmonic analysis on symmetric cones, and a division algorithm defined by the Cholesky decomposition to extend the definition of a beta-hypergeometric distribution to the cone of positive definite symmetric matrices generalizing the definition of a beta distribution on matrices (see [5]). These distributions are then used to extend to symmetric matrices the results established in the real case by Asci, Letac and Piccioni. It is worth mentioning here that in a private communication, Letac has given a definition of a beta-hypergeometric distribution on symmetric matrices using a division algorithm based on the notion of quadratic representation. The use of the division algorithm defined by the Cholesky decomposition is crucial in our

work, it allows the calculation of the spherical Fourier transform which the key tool in some proofs. The paper has the following plan. In Section 2, after a review of some facts concerning the beta distributions on symmetric matrices, we establish the matrix versions of the properties (1.1) and (1.2). We then introduce the matrix beta-hypergeometric distribution and we show some preliminary properties concerning this distribution. In Section 3, we state and prove our main results, in particular we use the matrix beta-hypergeometric distribution to give a matrix version of (1.3) and (1.4).

2 Matrix variate beta-hypergeometric distribution

Let V be the linear space of symmetric $r\times r$ matrices on $\mathbb{R}$, and Ω be the cone of positive definite elements of V. We denote the identity matrix by e, the determinant of an element x of V by $\Delta(x)$ and its trace by $\mathrm{tr}x$. We equip V with the inner product $\langle x,y\rangle=\mathrm{tr}(xy)$, for all $x,y\in V$. For an invertible $r\times r$ matrix a, we consider the automorphism $g(a)$ of V defined by $g(a)x=axa^*$ where a^* is the transpose of a. We denote G the group of such isomorphisms, K the subgroup of elements of G corresponding to a orthogonal, called the orthogonal group. As mentioned above, we will use the division algorithm on matrices based on the Cholesky decomposition of an element y of Ω, that is on the fact that y can be written in a unique manner as $y=tt^*$, where t is a lower triangular matrix with strictly positive diagonal (see [2]). For an element x in V, we set $\pi(y)(x)=txt^*$, and we define the "quotient" of x by y as $\pi^{-1}(y)(x)=t^{-1}x(t^*)^{-1}$, which, for simplicity, we sometimes denote abusively $\frac{x}{y}$.
Consider the absolutely continuous Wishart distribution concentrated on Ω with shape parameter $p>(r-1)/2$ and scale parameter $\sigma\in\Omega$,

$$W_{p,\sigma}(dx)=\frac{(\Delta(\sigma))^{-p}}{\Gamma_\Omega(p)}\exp(-\mathrm{tr}(x\sigma^{-1}))(\Delta(x))^{p-\frac{r+1}{2}}\mathbf{1}_\Omega(x)dx,$$

where

$$\Gamma_\Omega(p)=(2\pi)^{\frac{r(r-1)}{4}}\prod_{k=1}^{r}\Gamma(p-(k-1)/2).$$

If U and V are two independent Wishart random matrices with the same scale parameter σ and respective shape parameters $p>\frac{r-1}{2}$ and $q>\frac{r-1}{2}$, then the random matrix $\pi^{-1}(U+V)(U)$ has the so called beta distribution (of the first kind) $\beta^{(1)}_{p,q}$ on $\Omega\cap(e-\Omega)$ given by

$$\beta^{(1)}_{p,q}(dx)=(B_\Omega(p,q))^{-1}(\Delta(x))^{p-\frac{r+1}{2}}(\Delta(e-x))^{q-\frac{r+1}{2}}\mathbf{1}_{\Omega\cap(e-\Omega)}(x)dx,$$

where the normalizing constant $B_\Omega(p,q)$ is the multivariate beta function defined by

$$B_\Omega(p,q)=\frac{\Gamma_\Omega(p)\Gamma_\Omega(q)}{\Gamma_\Omega(p+q)}.$$

We also have the beta distribution (of the second kind) $\beta^{(2)}_{p,q}$ on Ω given by

$$\beta^{(2)}_{p,q}(dx)=(B_\Omega(p,q))^{-1}(\Delta(x))^{p-\frac{r+1}{2}}(\Delta(e+x))^{-(p+q)}\mathbf{1}_\Omega(x)dx.$$

It is the distribution of the random matrix $\pi^{-1}(V)(U)$, or equivalently the distribution of $\pi^{-1}(e-Z)(Z)$ with $Z\sim\beta^{(1)}_{p,q}$. More precisely, we have:

Proposition 2.1 *Let Y be a random matrix in Ω. Then $Y \sim \beta^{(2)}_{p,q}$ if and only if $Z = \pi^{-1}(e+Y)(Y) \sim \beta^{(1)}_{p,q}$.*

Proof Let $Y \sim \beta^{(2)}_{p,q}$ and $Z = \pi^{-1}(e+Y)(Y)$ which is equivalent to $Y = \pi^{-1}(e-Z)(Z)$. For a bounded measurable function h, we have

$$\begin{aligned}
E(h(Z)) &= (B_\Omega(p,q))^{-1}\int_\Omega h(\pi^{-1}(e+y)(y))\Delta(y)^{p-\frac{r+1}{2}}\Delta(e+y)^{-(p+q)}dy \\
&= (B_\Omega(p,q))^{-1}\int_{\Omega\cap(e-\Omega)} h(z)\Delta(\pi^{-1}(e-z)(z))^{p-\frac{r+1}{2}}\Delta(\pi^{-1}(e-z)(e))^{-(p+q)}\Delta(e-z)^{-(r+1)}dz \\
&= (B_\Omega(p,q))^{-1}\int_{\Omega\cap(e-\Omega)} h(z)\Delta(z)^{p-\frac{r+1}{2}}\Delta(e-z)^{q-\frac{r+1}{2}}dz.
\end{aligned}$$

Thus $Z \sim \beta^{(1)}_{p,q}$.
In a same way, we verify that if $Z \sim \beta^{(1)}_{p,q}$ then $\pi^{-1}(e-Z)(Z) \sim \beta^{(2)}_{p,q}$. □

Now, we give the matrix versions of (1.1) and (1.2).

Theorem 2.1 *Let W', X and W be three independent random matrices valued in Ω.*

$$i)\ \text{If}\ \ W' \sim \beta^{(2)}_{a+a',a'}\ \ \text{and}\ \ X \sim \beta^{(1)}_{a,a'},\ \ \text{then} \tag{2.5}$$
$$\pi^{-1}(e+\pi(X)(W'))(e) \sim \beta^{(1)}_{a',a}.$$
$$ii)\ \text{If}\ \ W \sim \beta^{(2)}_{a+a',a},\ X \sim \beta^{(1)}_{a,a'}\ \ \text{and}\ \ W' \sim \beta^{(2)}_{a+a',a'}\ \ ,\text{then} \tag{2.6}$$
$$\pi^{-1}(e+\pi^{-1}(e+\pi(X)(W'))(W))(e) \sim X.$$

Proof i) Let W' and X be two independent random matrix such that $W' \sim \beta^{(2)}_{a+a',a'}$ and $X \sim \beta^{(1)}_{a,a'}$. It is known (see Theorem 2.2 in [6]), that $\pi(X)(W') \sim \beta^{(2)}_{a,a'}$, and according to Proposition 2.1, we obtain that $\pi^{-1}(e+\pi(X)(W'))(\pi(X)(W')) \sim \beta^{(1)}_{a,a'}$. It follows that

$$\pi^{-1}(e+\pi(X)(W'))(e) = e - [\pi^{-1}(e+\pi(X)(W'))(\pi(X)(W'))] \sim \beta^{(1)}_{a',a}.$$

ii) As $\pi^{-1}(e+\pi(X)(W'))(W) = \pi(\pi^{-1}(e+\pi(X)(W')(e))(W)$, then according to Theorem 2.2 in [6] and to (2.5), we obtain that $\pi^{-1}(e+\pi(X)(W'))(W) \sim \beta^{(2)}_{a',a}$.
Therefore $\pi^{-1}(e+\pi^{-1}(e+\pi(X)(W'))(W))(e) \sim X$. □

In what follows, we will be interested in the extension of the results in Theorem 2.1 to the case where the parameter $a+a'$ in the distributions of W and W' is replaced by any parameter b not necessary equal to $a+a'$. For this we require some further terminology. Let $\mathcal{P}$ denote the space of polynomials on the space V of symmetric $r\times r$ matrices. A natural representation $\mathcal{H}$ of the group of automorphisms of V is defined for g in this group and p in $\mathcal{P}$ by $(\mathcal{H}(g)p)(x) = p(g^{-1}x)$.
For $X = (X_{ij})_{1\le i,j\le r}$ in Ω and $1 \le k \le r$, let $\Delta_k(X)$ denote the principal minor of order k of X, that is the determinant of the sub-matrix $P_k(X) = ((X_{ij})_{1\le i,j\le k})$. The generalized power of X is defined for $s = (s_1, ..., s_r) \in \mathbb{C}^r$, by

$$\Delta_s(X) = (\Delta_1(X))^{s_1-s_2}(\Delta_2(X))^{s_2-s_3}...(\Delta_r(X))^{s_r}.$$

For a given $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$ which satisfies $m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$ (denoted by $m \geq 0$), we denote by $\mathcal{P}_m$ the subspace of $\mathcal{P}$ generated by the polynomials $\mathcal{H}(g)\Delta_m$ where $g \in G$. The spherical polynomial ϕ_m is defined in [4] by

$$\phi_m(x) = \int_K \Delta_m(kx)dk.$$

Up to a constant factor, the ϕ_m are the only K-invariant polynomials in $\mathcal{P}_m$.
The definition of the beta-hypergeometric distribution on the cone of positive definite symmetric matrices relies on the notion of hypergeometric function which appears in [4], page 318. For instance, for $a = (a_1, \ldots, a_r)$ in $\mathbb{C}^r$ and $m = (m_1, \ldots, m_r)$ in $\mathbb{N}^r$ such that $m \geq 0$, we write

$$(a)_m = \frac{\Gamma_\Omega(a+m)}{\Gamma_\Omega(a)} = \prod_{i=1}^{r}(a_i - \frac{i-1}{2})_{m_i},$$

and for $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^r)$ in $\mathbb{C}^r$, $i = 1, \ldots, p$ and $\beta_j = (\beta_j^1, \ldots, \beta_j^r)$ in $\mathbb{C}^r$, $j = 1, \ldots, q$, we define the hypergeometric function

$$_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; x) = \sum_{m \geq 0} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{1}{(\frac{n}{r})_m} d_m \phi_m(x), \tag{2.7}$$

where d_m is the dimension of $\mathcal{P}_m$. Note that we can define the hypergeometric function for some complex α_i and β_j, $i = 1, \ldots, p$, $j = 1, \ldots, q$, where$(\alpha_i)_m = \prod_{j=1}^{r}(\alpha_i - \frac{j-1}{2})_{m_j}$ and $(\beta_j)_m = \prod_{i=1}^{r}(\beta_j - \frac{i-1}{2})_{m_i}$.
It is shown in [4], page 318, that the domain $\mathcal{D}$ of convergence of this series is:

- V, if $p \leq q$,
- $D = \{w; |w| < 1\}$ where $|.|$ is the spectral norm, if $p = q + 1$,
- $\emptyset$, if $p > q + 1$.

As it is done for the real beta-hypergeometric distribution (see[1]), we will be interested in the case $p = q + 1$. We will first show that under some conditions, the definition of $_{q+1}F_q(x)$ may be extended to $x = e$. This is in fact equivalent to show that the series (2.7) converges when $x = e$.

Proposition 2.2 *Let $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^r)$ and $\beta_j = (\beta_j^1, \ldots, \beta_j^r)$ in $\mathbb{R}^r$, such that $\alpha_j^i \neq \frac{i-1}{2}$ and $\beta_j^i \neq \frac{i-1}{2}$ for all $i = 1, \ldots, r$. Denote for $i = 1, \ldots, r$,*

$$c_i = \sum_{1 \leq j \leq q} \beta_j^i - \sum_{1 \leq j \leq q+1} \alpha_j^i.$$

Then the series

$$\sum_{m \geq 0} \frac{(\alpha_1)_m \ldots (\alpha_{q+1})_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{1}{(\frac{n}{r})_m} d_m, \tag{2.8}$$

converges if and only if, for all $1 \leq k \leq r$,

$$\sum_{i=1}^{k} c_i > 1 + k(r-k) - \frac{k(r+1)}{2}.$$

Note that in particular $c_1 > \frac{r-1}{2}$, and when $r = 1$, the condition reduces to $c_1 > 0$.
Proof Let $\alpha_j = (\alpha_j^1, \dots, \alpha_j^r) \in \mathbb{R}^r$ and $\beta_j = (\beta_j^1, \dots, \beta_j^r)$ in $\mathbb{R}^r$. We will consider separately two cases:

- Case where $\alpha_j^i > \frac{i-1}{2}$ and $\beta_j^i > \frac{i-1}{2}$ for all $i = 1, \dots, r$.
  Denote $p_i = m_i - m_{i+1}$, for $i = 1, \dots, r$, where $m_{r+1} = 0$. Then $(p_1, \dots p_r) \in \mathbb{N}^r$ and $m_i = \sum_{k=i}^r p_k$.
  Using the fact that
  $$d_m \simeq \prod_{1 \le i < j \le r} (1 + m_i - m_j),$$
  (see [4], page 286), we obtain that
  $$d_m \simeq \prod_{1 \le i < j \le r} (1 + \sum_{k=i}^{j-1} p_k).$$
  On the other hand, by Stirling approximation, we have that for $m_i \neq 0$,
  $$(\alpha_j^i - \frac{i-1}{2})_{m_i} \sim \frac{m_i^{(\alpha_j^i - \frac{i+1}{2})} m_i!}{\Gamma(\alpha_j^i - \frac{i-1}{2})}.$$
  Then
  $$(\alpha_j)_m \sim \prod_{i=1}^r \frac{m_i^{(\alpha_j^i - \frac{i+1}{2})} m_i!}{\Gamma(\alpha_j^i - \frac{i-1}{2})}, \quad j = 1, \dots, q+1.$$
  Consequently, the term of the series (2.8) is equivalent to
  $$A_1 A_2 \dots A_r \prod_{i=1}^r (\sum_{k=i}^r p_k)^{-c_i - \frac{n}{r}} \prod_{1 \le i < j \le r} (1 + \sum_{k=i}^{j-1} p_k),$$
  where $A_i = \frac{\Gamma(\frac{n}{r} - \frac{i-1}{2}) \prod_{k=1}^q \Gamma(\beta_k^i - \frac{i-1}{2})}{\prod_{k=1}^{q+1} \Gamma(\alpha_k^i - \frac{i-1}{2})}$ and $c_i = \sum_{1 \le j \le q} \beta_j^i - \sum_{1 \le j \le q+1} \alpha_j^i$, $i = 1, \dots, r$.
  Hence, the series (2.8) converges if and only if for all $1 \le k \le r$,
  $$\sum_{i=1}^k c_i > 1 + k(r-k) - \frac{k(r+1)}{2} \quad .$$

- Case where $\alpha_j^i < \frac{i-1}{2}$ or $\beta_j^i < \frac{i-1}{2}$ for some $i = 1, \dots, r$. There exists $k \in \mathbb{N}$ such that $-k < \alpha_j^i - \frac{i-1}{2} < -k+1$. This implies that $(\alpha_j^i - \frac{i-1}{2})_{m_i} = (\alpha_j^i - \frac{i-1}{2})_k (\alpha_j^i - \frac{i-1}{2} + k)_{m_i - k}$ for $m_i \ge k$. Then also the series (2.8) is convergent if and only if $\sum_{i=1}^k c_i > 1 + k(r-k) - \frac{k(r+1)}{2}$ for all $1 \le k \le r$.

$\square$

Note that if $\alpha_j^i = \frac{i-1}{2}$, for some $i = 1, ..., r$, then in the case where $m_i = 0$, $(\alpha_j)_m = \prod_{k=1, k \neq i}^r (\alpha_j^k - \frac{k-1}{2})$. If not $(\alpha_j)_m = 0$.
Hence the series (2.8) is convergent if and only if
$$\sum_{j=1}^k c_j > 1 + k(r-k) - \frac{k(r+1)}{2} \quad \text{for all} \quad 1 \le k \le i-1.$$

We are now in position to introduce the beta-hypergeometric distribution on symmetric matrices.

Definition 2.1 *The beta-hypergeometric distribution, with parameters* $(a, a', b) \in (]\frac{r-1}{2}, +\infty[)^3$, *is defined on* $\Omega \cap (e - \Omega)$ *by*

$$\mu_{a,a',b}(dx) = C(a, a', b)\Delta(x)^{a-\frac{n}{r}}\Delta(e - x)^{b-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; x)\boldsymbol{1}_{\Omega\cap(e-\Omega)}(x)(dx), \quad (2.9)$$

where

$$C(a, a', b) = \frac{\Gamma_\Omega(a + b)}{\Gamma_\Omega(a)\Gamma_\Omega(b)\ {}_3F_2(a, a, b; a + b, a + a'; e)}.$$

Note that the distribution $\mu_{a,a',a+a'}$ is nothing but the distribution $\beta^{(1)}_{a,a'}$. In fact, since we have

$${}_2F_1(a, b; a + a'; x) = \Delta(e - x)^{a'-b}\ {}_2F_1(a', a + a' - b; a + a'; x), \quad (2.10)$$

(see [4], page 330), then (2.9) becomes

$$\mu_{a,a',b}(dx) = \frac{\Gamma_\Omega(a + b)\Gamma_\Omega(a')}{\Gamma_\Omega(a + a')\Gamma_\Omega(b)\ {}_3F_2(a, a, b; a + b, a + a'; e)}\ {}_2F_1(a', a+a'-b; a+a'; x)\beta^{(1)}_{a,a'}(dx). \quad (2.11)$$

When $a + a' - b = 0$, ${}_3F_2(a, a, b; a + b, a + a'; e)$ becomes ${}_2F_1(a, a; 2a + a'; e)$. This, using the following Gauss formula

$${}_2F_1(\alpha, \beta; \gamma; e) = \frac{\Gamma_\Omega(\gamma)\Gamma_\Omega(\gamma - \alpha - \beta)}{\Gamma_\Omega(\gamma - \beta)\Gamma_\Omega(\gamma - \alpha)}, \quad (2.12)$$

for $\alpha = \beta = a$ and $\gamma = 2a + a'$ shows that $\mu_{a,a',a+a'}$ coincides with $\beta^{(1)}_{a,a'}$.
Next, we calculate the spherical Fourier transform of a beta-hypergeometric distribution, it is the expectation of its generalized power. This transform is important, it plays, for the K-invariant distributions on symmetric matrices, the role that the Mellin transform plays in the real case.

Proposition 2.3 *Let* X *be a random variable having the beta-hypergeometric distribution* $\mu_{a,a',b}$ *defined by (2.9). Then for* $t = (t_1, \dots, t_r) \in I\!R^r$ *such that* $t_i + a > \frac{i-1}{2}$, *for all* $1 \le i \le r$, *the spherical Fourier transform of* X *is*

$$E(\Delta_t(X)) = \frac{\Gamma_\Omega(a + b)}{\Gamma_\Omega(a)}\frac{\Gamma_\Omega(t + a)}{\Gamma_\Omega(t + a + b)}\frac{{}_3F_2(a, b, a + t; a + a', t + a + b; e)}{{}_3F_2(a, a, b; a + b, a + a'; e)}. \quad (2.13)$$

Proof

$$E(\Delta_t(X)) \;=\; C(a, a', b)\int_{\Omega\cap(e-\Omega)} \Delta_t(x)\Delta(x)^{a-\frac{n}{r}}\Delta(e - x)^{b-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; x)dx.$$

Since the determinant and the hypergeometric function are K-invariant, for $k \in K$,

$$E(\Delta_t(X)) \;=\; C(a, a', b)\int_{\Omega\cap(e-\Omega)} \Delta_t(x)\Delta(k^{-1}x)^{a-\frac{n}{r}}\Delta(e - k^{-1}x)^{b-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; k^{-1}x)dx.$$

With the change of variable $y = k^{-1}x$, we can write

$$\begin{aligned}
E(\Delta_t(X)) &= C(a,a',b)\int_{\Omega\cap(e-\Omega)} \Delta_t(ky)\Delta(y)^{a-\frac{n}{r}}\Delta(e-y)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';y)dy \\
&= C(a,a',b)\int_{\Omega\cap(e-\Omega)} \Delta_t(ky)\Delta(ky)^{a-\frac{n}{r}}\Delta(e-ky)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';y)dy \\
&= C(a,a',b)\sum_{m\geq 0}\frac{(a)_m(b)_m d_m}{(a+a')_m(\frac{n}{r})_m}\int_{\Omega\cap(e-\Omega)}\int_K \Delta_t(ky)\Delta(ky)^{a-\frac{n}{r}}\Delta(e-ky)^{b-\frac{n}{r}}\Delta_m(ky)dkdy \\
&= C(a,a',b)\sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\int_{\Omega\cap(e-\Omega)}\Delta_{m+t+a-\frac{n}{r}}(y)\Delta(e-y)^{b-\frac{n}{r}}dy \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(a+b)}{\Gamma_\Omega(a)\Gamma_\Omega(b)\ \ {}_3F_2(a,a,b;a+b,a+a';e)}\frac{\Gamma_\Omega(m+t+a)\Gamma_\Omega(b)}{\Gamma_\Omega(m+t+a+b)} \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m(t+a)_m}{(a+a')_m(t+a+b)_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(a+b)}{\Gamma_\Omega(a)\ \ {}_3F_2(a,a,b;a+b,a+a';e)}\frac{\Gamma_\Omega(t+a)}{\Gamma_\Omega(t+a+b)} \\
&= \frac{\Gamma_\Omega(a+b)}{\Gamma_\Omega(a)}\frac{\Gamma_\Omega(t+a)}{\Gamma_\Omega(t+a+b)}\frac{{}_3F_2(a,b,a+t;a+a',t+a+b;e)}{{}_3F_2(a,a,b;a+b,a+a';e)}.
\end{aligned}$$

□

3 Characterizations of the beta-hypergeometric distributions

In this section, we state and prove our main results involving the beta-hypergeometric probability measure $\mu_{a,a',b}$.

Theorem 3.1 *Let X and W be two independent random matrices such that $W \sim \beta^{(2)}_{b,a'}$ and $X \sim \mu_{a,a',b}$. Then*

$$\pi^{-1}(e+\pi(X)(W))(e) \sim \mu_{a',a,b}. \tag{3.14}$$

For the proof, we need to establish the following technical result.

Proposition 3.1 *For a, a', $b \in]\frac{r-1}{2},\infty[$ and $z \in \Omega\cap(e-\Omega)$, we have*

$$\int_{\Omega\cap(e-\Omega)}\frac{\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(t))^{a'+b}}\ {}_2F_1(a,b;a+a';e-t)dt = \frac{\Gamma_\Omega(a')\Gamma_\Omega(b)}{\Gamma_\Omega(a'+b)}\ {}_2F_1(a',b;a+a';z). \tag{3.15}$$

Proof We again use the invariance by the orthogonal group K of the determinant and of the hypergeometric function. For $k \in K$, we have

$$\begin{aligned}
I &= \int_{\Omega\cap(e-\Omega)}\frac{\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(t))^{a'+b}}\ {}_2F_1(a,b;a+a';e-t)dt \\
&= \int_{\Omega\cap(e-\Omega)}\frac{\Delta(e-k^{-1}t)^{a+a'-\frac{n}{r}}\Delta(k^{-1}t)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(t))^{a'+b}}\ {}_2F_1(a,b;a+a';e-k^{-1}t)dt.
\end{aligned}$$

Setting $y = k^{-1}t$ in last integral, we get

$$
\begin{aligned}
I &= \int_{\Omega\cap(e-\Omega)} \frac{\Delta(e-y)^{a+a'-\frac{n}{r}}\Delta(y)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(ky))^{a'+b}}\ {}_2F_1(a,b;a+a';e-y)dy \\
&= \int_{\Omega\cap(e-\Omega)} \frac{\Delta(e-ky)^{a+a'-\frac{n}{r}}\Delta(ky)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(ky))^{a'+b}}\ {}_2F_1(a,b;a+a';e-y)dy \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\int_{\Omega\cap(e-\Omega)}\int_K \frac{\Delta(e-ky)^{a+a'-\frac{n}{r}}\Delta(ky)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(ky))^{a'+b}}\Delta_m(e-ky)dkdy \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\int_{\Omega\cap(e-\Omega)} \frac{\Delta(e-y)^{a+a'-\frac{n}{r}}\Delta(y)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(y))^{a'+b}}\Delta_m(e-y)dy \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\int_{\Omega\cap(e-\Omega)} \frac{\Delta_{m+a+a'-\frac{n}{r}}(e-y)\Delta(y)^{b-\frac{n}{r}}}{\Delta(e-\pi(z)(y))^{a'+b}}dy \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(b)\Gamma_\Omega(m+a+a')}{\Gamma_\Omega(m+a+a'+b)}\ {}_2F_1(a'+b,b;b+m+a+a';z) \\
&= \sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(b)\Gamma_\Omega(m+a+a')}{\Gamma_\Omega(m+a+a'+b)}\sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(b+m+a+a')_k(\frac{n}{r})_k}d_k\phi_k(z) \\
&= \sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(\frac{n}{r})_k}d_k\phi_k(z)\sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(b)\Gamma_\Omega(m+a+a')}{\Gamma_\Omega(m+a+a'+b)(b+m+a+a')_k} \\
&= \sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(\frac{n}{r})_k}d_k\phi_k(z)\sum_{m\geq 0}\frac{(a)_m(b)_m d_m}{(a+a')_m(\frac{n}{r})_m}\frac{\Gamma_\Omega(b)\Gamma_\Omega(m+a+a')}{\Gamma_\Omega(m+a+a'+b)}\frac{\Gamma_\Omega(b+a+a'+m)}{\Gamma_\Omega(b+m+a+a'+k)} \\
&= \sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(\frac{n}{r})_k}d_k\phi_k(z)\sum_{m\geq 0}\frac{(a)_m(b)_m d_m}{(b+a+a'+k)_m(\frac{n}{r})_m}\frac{\Gamma_\Omega(b)\Gamma_\Omega(a+a')}{\Gamma_\Omega(b+a+a'+k)} \\
&= \sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(\frac{n}{r})_k}d_k\phi_k(z)\frac{\Gamma_\Omega(b)\Gamma_\Omega(a+a')}{\Gamma_\Omega(b+a+a'+k)}\ {}_2F_1(a,b;a+a'+b+k;e) \\
&= \sum_{k\geq 0}\frac{(a'+b)_k(b)_k}{(\frac{n}{r})_k}d_k\phi_k(z)\frac{\Gamma_\Omega(b)\Gamma_\Omega(a+a')}{\Gamma_\Omega(b+a+a'+k)}\frac{\Gamma_\Omega(a+a'+b+k)\Gamma_\Omega(a'+k)}{\Gamma_\Omega(a+a'+k)\Gamma_\Omega(a'+b+k)} \\
&= \sum_{k\geq 0}\frac{(b)_k(a')_k}{(a+a')_k(\frac{n}{r})_k}d_k\phi_k(z)\frac{\Gamma_\Omega(b)\Gamma_\Omega(a')}{\Gamma_\Omega(a'+b)} \\
&= \frac{\Gamma_\Omega(b)\Gamma_\Omega(a')}{\Gamma_\Omega(a'+b)}\ {}_2F_1(a',b;a+a';z).
\end{aligned}
$$

□

We come now to the proof of Theorem 3.1.

Proof of Theorem 3.1 Let X' be a random variable with distribution $\mu_{a',a,b}$, and define $V = \pi^{-1}(X')(e-X')$. Then showing (3.14) is equivalent to show that V and $\pi(X)(W)$ have the same distribution. Let h be a bounded measurable function. Then

$$
\begin{aligned}
E(h(V)) &= \int_{\Omega\cap(e-\Omega)} h(\pi^{-1}(x)(e-x))\mu_{a',a,b}(dx) \\
&= C(a',a,b)\int_{\Omega\cap(e-\Omega)} h(\pi^{-1}(x)(e-x))\Delta(x)^{a'-\frac{n}{r}}\Delta(e-x)^{b-\frac{n}{r}}\ {}_2F_1(a',b;a+a';x)dx.
\end{aligned}
$$

Setting $y = \pi^{-1}(x)(e-x)$, or equivalently $x = \pi^{-1}(e+y)(e)$, then $dx = \Delta(e+y)^{-\frac{2n}{r}}dy$, and we have

$$\begin{aligned} E(h(V)) &= C(a',a,b)\int_\Omega h(y)\Delta(\pi^{-1}(e+y)(e))^{a'-\frac{n}{r}}\Delta(e-\pi^{-1}(e+y)(e))^{b-\frac{n}{r}} \\ &\quad {}_2F_1(a',b;a+a';\pi^{-1}(e+y)(e))\Delta(e+y)^{-\frac{2n}{r}}dy \\ &= C(a',a,b)\int_\Omega h(y)\Delta(\pi^{-1}(e+y)(e))^{a'+b}\Delta(y)^{b-\frac{n}{r}}\ {}_2F_1(a',b;a+a';\pi^{-1}(e+y)(e))dy. \end{aligned}$$

Hence the density of V is

$$f_V(v) = C(a',a,b)\Delta(\pi^{-1}(e+v)(e))^{a'+b}\Delta(v)^{b-\frac{n}{r}}\ {}_2F_1(a',b;a+a';\pi^{-1}(e+v)(e))\mathbf{1}_\Omega(v). \tag{3.16}$$

On the other hand, the density of $U = \pi(X)(W)$ is given by

$$\begin{aligned} f_U(u) &= \int_{\Omega\cap(e-\Omega)} f_X(x)f_W(\pi^{-1}(x)(u))\Delta(x)^{-\frac{n}{r}}dx \\ &= C(a,a',b)\frac{\Gamma_\Omega(a'+b)}{\Gamma_\Omega(a')\Gamma_\Omega(b)}\int_{\Omega\cap(e-\Omega)}\Delta(x)^{a-\frac{n}{r}}\Delta(e-x)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';x)\Delta(\pi^{-1}(x)(u))^{b-\frac{n}{r}} \\ &\quad \Delta(e+\pi^{-1}(x)(u))^{-b-a'}\Delta(x)^{-\frac{n}{r}}dx \\ &= C(a,a',b)\frac{\Gamma_\Omega(a'+b)}{\Gamma_\Omega(a')\Gamma_\Omega(b)}\Delta(u)^{b-\frac{n}{r}}\int_{\Omega\cap(e-\Omega)}\Delta(x)^{a+a'-\frac{n}{r}}\Delta(e-x)^{b-\frac{n}{r}}\Delta(x+u)^{-b-a'} \\ &\quad {}_2F_1(a,b;a+a';x)dx. \end{aligned}$$

With the change $t = e-x$, we get

$$\begin{aligned} f_U(u) &= C(a,a',b)\frac{\Gamma_\Omega(a'+b)}{\Gamma_\Omega(a')\Gamma_\Omega(b)}\Delta(u)^{b-\frac{n}{r}}\int_{\Omega\cap(e-\Omega)}\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}\Delta(e+u-t)^{-b-a'} \\ &\quad {}_2F_1(a,b;a+a';e-t)dt \\ &= C(a,a',b)\frac{\Gamma_\Omega(a'+b)}{\Gamma_\Omega(a')\Gamma_\Omega(b)}\frac{\Delta(u)^{b-\frac{n}{r}}}{\Delta(e+u)^{a'+b}}\int_{\Omega\cap(e-\Omega)}\frac{\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}}{\Delta(e-\pi^{-1}(e+u)(t))^{b+a'}} \\ &\quad {}_2F_1(a,b;a+a';e-t)dt. \end{aligned}$$

Using the fact that $\pi^{-1}(e+u)(t) = \pi(\pi^{-1}(e+u)(e))(t)$, and invoking Proposition 3.1, we obtain that

$$\begin{aligned} &\int_{\Omega\cap(e-\Omega)}\frac{\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}}{\Delta(e-\pi(\pi^{-1}(e+u)(e))(t))^{a'+b}}\ {}_2F_1(a,b;a+a';e-t)dt \\ &= \int_{\Omega\cap(e-\Omega)}\Delta(e-t)^{a+a'-\frac{n}{r}}\Delta(t)^{b-\frac{n}{r}}\Delta(e-\pi(\pi^{-1}(e+u)(e))(t))^{-a'-b}\ {}_2F_1(a,b;a+a';e-t)dt \\ &= \frac{\Gamma_\Omega(b)\Gamma_\Omega(a')}{\Gamma_\Omega(a'+b)}\ {}_2F_1(a',b;a+a';\pi^{-1}(e+u)(e)). \end{aligned}$$

Consequently, the density of U is equal to

$$f_U(u) = C(a,a',b)\Delta(\pi^{-1}(e+u)(e))^{a'+b}\Delta(u)^{b-\frac{n}{r}}\ {}_2F_1(a',b;a+a';\pi^{-1}(e+u)(e))\mathbf{1}_\Omega(u). \tag{3.17}$$

Comparing (3.16) and (3.17), we conclude that the densities of U and V are equal, and consequently, their normalizing constants $C(a,a',b)$ and $C(a',a,b)$ are equal. $\square$

Note that the fact that $C(a,a',b)$ is a symmetric function of (a,a') means that

$$\frac{{}_3F_2(a,a,b;a+b,a+a';e)}{\Gamma_\Omega(a')\Gamma_\Omega(a+b)}=\frac{{}_3F_2(a',a',b;a'+b,a+a';e)}{\Gamma_\Omega(a)\Gamma_\Omega(a'+b)}. \tag{3.18}$$

We can deduce another expression of the spherical Fourier transform of the beta-hypergeometric distribution from the following more general result.

Proposition 3.1 *1. For $t=(t_1,\dots,t_r)\in I\!R^r$ and $s=(s_1,\dots,s_r)\in I\!R^r$, the integral*

$$I_{a,a',b}(t,s)=\int_{\Omega\cap(e-\Omega)}\Delta_t(x)\Delta_s(e-x)\mu_{a,a',b}(dx)$$

converges if and only if for $i=1,\dots,r$,

$$t_i>\frac{i-1}{2}-a,\quad s_i>\frac{i-1}{2}-b,$$

and for all $1\leq k\leq r$,

$$\sum_{i=1}^{k}s_i+ka'>1+k(r-k)-k\frac{(r+1)}{2}.$$

In this case, we have

$$I_{a,a',b}(t,s)=\frac{\Gamma_\Omega(a+b)\Gamma_\Omega(a+t)\Gamma_\Omega(b+s)}{\Gamma_\Omega(a)\Gamma_\Omega(b)\Gamma_\Omega(a+b+t+s)}\frac{{}_3F_2(a+t,a,b;a+b+t+s,a+a';e)}{{}_3F_2(a,a,b;a+b,a+a';e)}. \tag{3.19}$$

2. We also have under the conditions $\sum_{i=1}^{k}t_i+ka>1+k(r-k)-k\frac{(r+1)}{2}$, for all $1\leq k\leq r$,

$$I_{a,a',b}(t,0)=\int_{\Omega\cap(e-\Omega)}\Delta_t(x)\mu_{a,a',b}(dx)=\frac{{}_3F_2(a',a'-t,b;a'+b,a+a';e)}{{}_3F_2(a',a',b;a'+b,a+a';e)}. \tag{3.20}$$

Proof
1) For simplicity, we denote $C=C(a,a',b)$. We first calculate the integral

$$\begin{aligned}I_{a,a',b}(t,s)&=\int_{\Omega\cap(e-\Omega)}\Delta_t(x)\Delta_s(e-x)\mu_{a,a',b}(dx)\\&=C\int_{\Omega\cap(e-\Omega)}\Delta_t(x)\Delta_s(e-x)\Delta(x)^{a-\frac{n}{r}}\Delta(e-x)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';x)dx.\end{aligned}$$

As the determinant and the hypergeometric function are K- invariant, for $k\in K$, we have

$$I_{a,a',b}(t,s)=C\int_{\Omega\cap(e-\Omega)}\Delta_t(x)\Delta_s(e-x)\Delta(k^{-1}x)^{a-\frac{n}{r}}\Delta(e-k^{-1}x)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';k^{-1}x)dx.$$

Setting $y = k^{-1}x$, we get

$$\begin{aligned} I_{a,a',b}(t,s) &= C\int_{\Omega\cap(e-\Omega)} \Delta_t(ky)\Delta_s(e-ky)\Delta(y)^{a-\frac{n}{r}}\Delta(e-y)^{b-\frac{n}{r}}\ {}_2F_1(a,b;a+a';y)dy \\ &= C\int_{\Omega\cap(e-\Omega)} \Delta_t(ky)\Delta_s(e-ky)\Delta(ky)^{a-\frac{n}{r}}\Delta(e-ky)^{b-\frac{n}{r}}\sum_{m\geq 0}\frac{(a)_m(b)_m d_m}{(a+a')_m(\frac{n}{r})_m}\phi_m(y)dy. \end{aligned}$$

Since all terms are positive we can invert sums and integrals, whether they converge or not. Hence

$$I_{a,a',b}(t,s) = C\sum_{m\geq 0}\frac{(a)_m(b)_m d_m}{(a+a')_m(\frac{n}{r})_m}\int_{\Omega\cap(e-\Omega)}\int_K \Delta_t(ky)\Delta_s(e-ky)\Delta(ky)^{a-\frac{n}{r}}\Delta(e-ky)^{b-\frac{n}{r}}$$

$$\Delta_m(ky)dkdy.$$

It follows that

$$I_{a,a',b}(t,s) = C\sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\int_{\Omega\cap(e-\Omega)}\Delta_{m+a+t-\frac{n}{r}}(z)\Delta_{s+b-\frac{n}{r}}(e-z)dz. \tag{3.21}$$

This last integral converges if and only if $a+t_i > \frac{i-1}{2}$ and $b+s_i > \frac{i-1}{2}$, and under these conditions, it is equal to

$$C\sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(m+t+a)\Gamma_\Omega(s+b)}{\Gamma_\Omega(m+t+a+s+b)}.$$

Thus

$$\begin{aligned} I_{a,a',b}(t,s) &= C\sum_{m\geq 0}\frac{(a)_m(b)_m}{(a+a')_m(\frac{n}{r})_m}d_m\frac{\Gamma_\Omega(m+t+a)\Gamma_\Omega(s+b)}{\Gamma_\Omega(m+t+a+s+b)} \\ &= C\sum_{m\geq 0}\frac{(a)_m(b)_m(t+a)_m d_m}{(a+a')_m(t+a+s+b)_m(\frac{n}{r})_m}\frac{\Gamma_\Omega(t+a)\Gamma_\Omega(s+b)}{\Gamma_\Omega(t+a+s+b)}. \end{aligned}$$

From Proposition 2.2, this series converges if and only if for all $1\leq k\leq r$,

$$\sum_{i=1}^k s_i + ka' > 1 + k(r-k) - k\frac{(r+1)}{2},$$

and under this condition, we have that

$$I_{a,a',b}(t,s) = \frac{{}_3F_2(t+a,a,b;t+a+s+b,a+a';e)}{{}_3F_2(a,a,b;a+b,a+a';e)}\frac{\Gamma_\Omega(a+b)\Gamma_\Omega(t+a)\Gamma_\Omega(s+b)}{\Gamma_\Omega(a)\Gamma_\Omega(b)\Gamma_\Omega(t+a+s+b)}.$$

2) For this second part, we use Theorem 3.1. Consider two independent random variables $X\sim\mu_{a,a',b}$ and $W\sim\beta^{(2)}_{b,a'}$. Then $X' = \pi^{-1}(e+\pi(X)(W))(e)\sim\mu_{a',a,b}$.
Since $\pi^{-1}(X')(e-X') = \pi(X)(W)$ and $E(\Delta_t(W)) = \frac{\Gamma_\Omega(b+t)\Gamma_\Omega(a'-t)}{\Gamma_\Omega(b)\Gamma_\Omega(a')}$, we have

$$\begin{aligned} E(\Delta_t(\pi(X)(W))) &= E(\Delta_t(X))E(\Delta_t(W)) \\ &= E(\Delta_t(\pi^{-1}(X')(e-X'))) \\ &= E(\frac{1}{\Delta_t(X')}\Delta_t(e-X')). \end{aligned}$$

It follows that

$$
\begin{aligned}
E(\Delta_t(X)) &= \frac{1}{E(\Delta_t(W))}E(\frac{1}{\Delta_t(X')}\Delta_t(e-X')) \\
&= \frac{1}{E(\Delta_t(W))}E(\Delta_{-t}(X')\Delta_t(e-X')).
\end{aligned}
$$

Now we apply the first part of the proposition by replacing (a,a',b,t,s) by $(a',a,b,-t,t)$, getting the result for $\frac{i-1}{2}-b < t_i < a'-\frac{i-1}{2}$ and $\sum_{i=1}^k t_i + ka > 1+k(r-k)-k\frac{(r+1)}{2}$ for all $1 \le k \le r$. Under these conditions, we obtain that the spherical Fourier transform of X is

$$
E(\Delta_t(X)) = \frac{{}_3F_2(a',a'-t,b;a'+b,a+a';e)}{{}_3F_2(a',a',b;a'+b,a+a';e)}. \tag{3.22}
$$

Finally, we observe that the right hand side of (3.22) is finite if and only if for all $1 \le k \le r$, $\sum_{i=1}^k t_i + ka > 1+k(r-k)-k\frac{r+1}{2}$ and it is a positive analytic function of t satisfying this condition. The principle of maximal analyticity implies that (3.20) holds for t such that for all $1 \le k \le r$, $\sum_{i=1}^k t_i + ka > 1+k(r-k)-k\frac{(r+1)}{2}$.

$\square$

Remark 3.1 1. *From (3.19) and (3.20), we obtain two different expressions of $E(\Delta_t(X))$. Equating these expressions and recalling (3.18), we obtain the following relation concerning the function ${}_3F_2$.*

$$
\frac{{}_3F_2(a+t,a,b;t+a+b,a+a';e)}{\Gamma_\Omega(a')\Gamma_\Omega(a+b+t)} = \frac{{}_3F_2(a',a'-t,b;a'+b,a+a';e)}{\Gamma_\Omega(a+t)\Gamma_\Omega(a'+b)}, \tag{3.23}
$$

for $t_i > \frac{i-1}{2}-a$ and $\sum_{i=1}^k t_i + ka > 1+k(r-k)-k\frac{(r+1)}{2}$ for all $1 \le k \le r$.

2. *Using the characterization of the beta-hypergeometric distribution by its spherical Fourier transform given in (3.22), we can easily show the converse of Theorem 3.1, that is if X and W are two independent random matrices in Ω such that $W \sim \beta^{(2)}_{b,a'}$, then $\pi^{-1}(e+\pi(X)(W))(e) \sim \mu_{a',a,b}$ implies that $X \sim \mu_{a,a',b}$.*

Next, we give the matrix version of (1.4).

Theorem 3.2 1. *Let $W \sim \beta^{(2)}_{b,a}$, $W' \sim \beta^{(2)}_{b,a'}$ and X be three independent random variables, with X valued in Ω. Then*

$$
X \sim \pi^{-1}(e+\pi^{-1}(e+\pi(X)(W'))(W))(e) \quad \textit{if and only if} \quad X \sim \mu_{a,a',b} \tag{3.24}
$$

2. *If $W \sim \beta^{(2)}_{b,a}$ and $X \in \Omega$ are two independent random variables, then*

$$
X \sim \pi^{-1}(e+\pi(X)(W))(e) \quad \textit{if and only if} \quad X \sim \mu_{a,a,b} \tag{3.25}
$$

3. *Let* $(W_n)_{n\geq 1}$ *and* $(W'_n)_{n\geq 1}$ *be two independent sequences of random variables with respective distributions* $\beta^{(2)}_{b,a}$ *and* $\beta^{(2)}_{b,a'}$. *Then* $\mu_{a,a',b}$ *is the distribution of the random continued fraction*

$$\cfrac{e}{e+\cfrac{W_1}{e+\cfrac{W'_1}{e+\cfrac{W_2}{e+\cfrac{W'_2}{e+\dots}}}}}. \tag{3.26}$$

Proof We adapt the method of proof used in the real case by Asci, Letac and Piccioni [1] to the matrix case.

1. Observe first the series (3.26) converges almost surely, because the series $\sum_n (W_n^{-1} + W_n'^{-1})$ diverges almost surely. Consider the sequence $(F_n)_{n=1}^{\infty}$ of random mappings from $\Omega \cap (e-\Omega)$ into itself defined by

$$F_n(z) = \pi^{-1}(e + \pi^{-1}(e + \pi(z)(W'_n))(W_n))(e).$$

Since $F_1 \circ \ldots \circ F_n(z)$ has almost surely a limit X, then the distribution of X is a stationary distribution of the Markov chain $w_n = F_n \circ \ldots \circ F_1(z)$ which is unique. According to Theorem (3.1), we have that $\mu_{a,a',b}$ is a stationary distribution of the Markov chain $(w_n)_{n=0}^{\infty}$. It follows that $X \sim \mu_{a,a',b}$.

2. We use the reasoning above with the random mappings $G_n(z) = \pi^{-1}(e + \pi(z)(W_n))(e)$.

3. The proof of this part is similar to the first part.

$\square$

In the following theorem, we establish the identifiability of the beta-hypergeometric distribution on symmetric matrices.

Theorem 3.3 *Let* (a, a', b) *and* (a_1, a'_1, b_1) *in* $(]\frac{r-1}{2}, \infty[)^3$.

$$\textit{If}\ \ \mu_{a,a',b} = \mu_{a_1,a'_1,b_1},\ \ \textit{then}\ \ (a, a', b) = (a_1, a'_1, b_1).$$

Proof For the sake of simplification, we denote $C = C(a, a', b)$ and $C_1 = C(a_1, a'_1, b_1)$. For $x \in \Omega \cap (e - \Omega)$, we have that

$$\begin{aligned}
\lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}}\ {}_2F_1(a, b; a+a'; x) &= \lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}} \sum_{m\geq 0} \frac{(a)_m (b)_m}{(a+a')_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(x) \\
&= \lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}} d_{(0,\ldots,0)} \\
&+ \lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}} \sum_{m\geq 0; m\neq 0} \frac{(a)_m (b)_m}{(a+a')_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(x) \\
&= \lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}} d_{(0,\ldots,0)} \\
&+ \lim_{x\to 0} \sum_{m\geq 0; m\neq 0} \Delta(e-x)^{b-\frac{n}{r}} \frac{(a)_m (b)_m}{(a+a')_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(x).
\end{aligned}$$

Since $d_{(0,\dots,0)} = 1$, and

$$\lim_{x\to 0} \sum_{m\geq 0; m\neq 0} \Delta(e-x)^{b-\frac{n}{r}} \frac{(a)_m(b)_m}{(a+a')_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(x) = 0,$$

we conclude that

$$\lim_{x\to 0} \Delta(e-x)^{b-\frac{n}{r}} \;\; {}_2F_1(a,b;a+a';x) = 1.$$

Thus, when x is close to 0, the densities of $\mu_{a,a',b}$ and μ_{a_1,a'_1,b_1} are respectively equivalent to $C\Delta(x)^{a-\frac{n}{r}}$ and $C_1\Delta(x)^{a_1-\frac{n}{r}}$. Since $\mu_{a,a',b} = \mu_{a_1,a'_1,b_1}$, we get $C\Delta(x)^{a-\frac{n}{r}} = C_1\Delta(x)^{a_1-\frac{n}{r}}$. Hence $a = a_1$, and it follows that for all x in $\Omega \cap (e-\Omega)$,

$$\Delta(e-x)^{b-\frac{n}{r}} \;\; {}_2F_1(a,b;a+a';x) = \Delta(e-x)^{b_1-\frac{n}{r}} \;\; {}_2F_1(a,b_1;a+a'_1;x).$$

Using Proposition $XV.3.4$ page 330 in [4], we can write for x in $\Omega \cap (e-\Omega)$,

$${}_2F_1(a,b;a+a';x) = \Delta(e-x)^{-b} \;\; {}_2F_1(a',b;a+a';-x(e-x)^{-1})$$

and

$${}_2F_1(a,b_1;a+a'_1;x) = \Delta(e-x)^{-b_1} \;\; {}_2F_1(a'_1,b_1;a+a'_1;-x(e-x)^{-1}).$$

Therefore

$${}_2F_1(a',b;a+a';z) = \;\; {}_2F_1(a'_1,b_1;a+a'_1;z),$$

or equivalently

$$\sum_{m\geq 0} \frac{(a')_m(b)_m}{(c)_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(z) = \sum_{m\geq 0} \frac{(a'_1)_m(b_1)_m}{(c_1)_m} \frac{d_m}{(\frac{n}{r})_m} \phi_m(z)$$

where $c = a + a'$ and $c_1 = a + a'_1$.
Since $\phi_m(z)$ is a polynomial in z with degree equal to $|m| = m_1 + \ldots + m_r$, this implies that

$$\frac{(a')_m(b)_m}{(c)_m} \frac{d_m}{(\frac{n}{r})_m} = \frac{(a'_1)_m(b_1)_m}{(c_1)_m} \frac{d_m}{(\frac{n}{r})_m},$$

for each $m \geq 0$.
For $m = (1,0,\dots,0)$, we obtain

$$\frac{a'b}{c} = \frac{a'_1 b_1}{c_1}.$$

For $m = (1,1,0,\dots,0)$, we obtain

$$\frac{(a'-\frac{1}{2})(b-\frac{1}{2})}{(c-\frac{1}{2})} = \frac{(a'_1-\frac{1}{2})(b_1-\frac{1}{2})}{(c_1-\frac{1}{2})}.$$

Finally, for $m = (1,1,1,0,\dots,0)$, we obtain

$$\frac{(a'-1)(b-1)}{(c-1)} = \frac{(a'_1-1)(b_1-1)}{(c_1-1)}.$$

Let

$$\lambda_0 = \frac{a'b}{c}, \quad \lambda_1 = \frac{(a'-\frac{1}{2})(b-\frac{1}{2})}{(c-\frac{1}{2})} \quad \text{and} \quad \lambda_2 = \frac{(a'-1)(b-1)}{(c-1)}.$$

By taking suitable linear combination, we get

$$a'b = c\lambda_0,\ a' + b = 2c\lambda_0 - 2(c - \frac{1}{2})\lambda_1 + \frac{1}{2},\ c(\lambda_2 + \lambda_0 - 2\lambda_1) + \lambda_1 - \lambda_2 - \frac{1}{2} = 0.$$

From this, c can be uniquely determined, we get $c = c_1$ then $a' = a'_1$, and $b = b_1$. $\square$

Theorem 3.4 *Let X be a beta-hypergeometric random matrix, $X \sim \mu_{a,a',b}$. Then $(e - X) \sim \mu_{a_1,a'_1,b_1}$ if and only if $a_1 = a'$, $a = a'_1$ and $b_1 = b = a + a' = a_1 + a'_1$.*

Proof ($\Leftarrow$) This way is obvious.
($\Rightarrow$) Suppose that $X \sim \mu_{a,a',b}$ and $e - X \sim \mu_{a_1,a'_1,b_1}$. Since the beta-hypergeometric distribution is K-invariant, it is characterized by its spherical Fourier transform.

$$\begin{aligned} E(\Delta_t(e - X)) &= C(a, a', b) \int_{\Omega\cap(e-\Omega)} \Delta_t(e - x)\Delta(x)^{a-\frac{n}{r}}\Delta(e - x)^{b-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; x)dx \\ &= C(a, a', b) \int_{\Omega\cap(e-\Omega)} \Delta_{t+b-\frac{n}{r}}(e - x)\Delta(x)^{a-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; x)dx. \\ &= C(a, a', b) \int_{\Omega\cap(e-\Omega)} \Delta_{t+b-\frac{n}{r}}(e - x)\Delta(x)^{a-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; k^{-1}x)dx, \end{aligned}$$

where the last equality is due to the fact that the hypergeometric function is K-invariant. Setting $y = k^{-1}x$, we obtain that

$$\begin{aligned} E(\Delta_t(e - X)) &= C(a, a', b) \int_{\Omega\cap(e-\Omega)} \Delta_{t+b-\frac{n}{r}}(e - ky)\Delta(ky)^{a-\frac{n}{r}}\ {}_2F_1(a, b; a + a'; y)dy \\ &= C(a, a', b) \sum_{m\geq 0} \frac{(a)_m(b)_m}{(a + a')_m} \frac{d_m}{(\frac{n}{r})_m} \int_{\Omega\cap(e-\Omega)} \int_K \Delta_{t+b-\frac{n}{r}}(e - ky)\Delta_{m+a-\frac{n}{r}}(ky)dkdy \\ &= C(a, a', b) \sum_{m\geq 0} \frac{(a)_m(b)_m}{(a + a')_m} \frac{d_m}{(\frac{n}{r})_m} \int_{\Omega\cap(e-\Omega)} \Delta_{t+b-\frac{n}{r}}(e - y)\Delta_{m+a-\frac{n}{r}}(y)dy. \end{aligned}$$

The last integral converges when $t_i + b > \frac{i-1}{2}$ for all $1 \leq i \leq r$. Under this condition we can write that

$$\begin{aligned} E(\Delta_t(e - X)) &= C(a, a', b) \sum_{m\geq 0} \frac{(a)_m(b)_m}{(a + a')_m} \frac{d_m}{(\frac{n}{r})_m} \frac{\Gamma_\Omega(m + a)\Gamma_\Omega(t + b)}{\Gamma_\Omega(m + a + t + b)} \\ &= \sum_{m\geq 0} \frac{(a)_m(b)_m(a)_m}{(a + a')_m(a + t + b)_m} \frac{d_m}{(\frac{n}{r})_m} \frac{\Gamma_\Omega(a + b)\Gamma_\Omega(t + b)}{\Gamma_\Omega(b)\Gamma_\Omega(a + t + b)\ {}_3F_2(a, a, b; a + b, a + a'; e)} \\ &= \frac{\Gamma_\Omega(a + b)\Gamma_\Omega(t + b)}{\Gamma_\Omega(b)\Gamma_\Omega(a + t + b)} \frac{{}_3F_2(a, a, b; a + a', a + b + t; e)}{{}_3F_2(a, a, b; a + b, a + a'; e)}, \end{aligned}$$

which is defined for t such that $\sum_{i=1}^k t_i + ka' > 1 + k(r - k) - \frac{k(r+1)}{2}$ for all $1 \leq k \leq r$. Also since $e - X \sim \mu_{a_1,a'_1,b_1}$, we use (2.13) for t such that for $1 \leq i \leq r$, $t_i + a_1 > \frac{i-1}{2}$, to obtain that

$$\begin{aligned} &\frac{\Gamma_\Omega(a + b)\Gamma_\Omega(t + b)}{\Gamma_\Omega(b)\Gamma_\Omega(a + t + b)} \frac{{}_3F_2(a, a, b; a + a', a + b + t; e)}{{}_3F_2(a, a, b; a + b, a + a'; e)} \\ &= \frac{\Gamma_\Omega(a_1 + b_1)\Gamma_\Omega(t + a_1)}{\Gamma_\Omega(a_1)\Gamma_\Omega(a_1 + t + b_1)} \frac{{}_3F_2(a_1, b_1, a_1 + t; a_1 + a'_1, a_1 + b_1 + t; e)}{{}_3F_2(a_1, a_1, b_1; a_1 + b_1, a_1 + a'_1; e)}. \end{aligned}$$

The last equality is equivalent to

$$\frac{{}_3F_2(a_1,b_1,a_1+t;a_1+a_1',a_1+b_1+t;e)}{{}_3F_2(a,a,b;a+a',a+b+t;e)}$$

$$=\frac{\Gamma_\Omega(a+b)\Gamma_\Omega(a_1)}{\Gamma_\Omega(b)\Gamma_\Omega(a_1+b_1)}\frac{{}_3F_2(a_1,a_1,b_1;a_1+b_1,a_1+a_1';e)}{{}_3F_2(a,a,b;a+b,a+a';e)}\frac{\Gamma_\Omega(t+b)\Gamma_\Omega(a_1+t+b_1)}{\Gamma_\Omega(a+t+b)\Gamma_\Omega(t+a_1)}.$$

Hence the function $t\mapsto \frac{{}_3F_2(a_1,b_1,a_1+t;a_1+a_1',t+a_1+b_1;e)}{{}_3F_2(a,a,b;a+a',t+a+b;e)}$ is expressed in terms of gamma functions. This happens if and only if $b_1=a_1+a_1'$ and $b=a+a'$. In this case we have that $e-X\sim\beta^{(1)}_{a_1,a_1'}$ and $X\sim\beta^{(1)}_{a,a'}$. However when $X\sim\beta^{(1)}_{a,a'}$, we have that $e-X\sim\beta^{(1)}_{a',a}$. Hence $a_1=a'$ and $a_1'=a$. □

Proposition 3.2 *The following convergences in law hold:*

1. $\lim_{a\to 0}\mu_{a,a',b}=\delta_0$ *and* $\lim_{a\to\infty}\mu_{a,a',b}=\delta_e$,
2. $\lim_{a'\to 0}\mu_{a,a',b}=\delta_e$ *and* $\lim_{a'\to\infty}\mu_{a,a',b}=\beta^{(1)}_{a,b}$,
3. $\lim_{b\to 0}\mu_{a,a',b}=\delta_e$, $\lim_{b\to\infty}\mu_{a,a',b}=\delta_0$ *if* $a-\frac{r-1}{2}\le a'$ *and* $\lim_{b\to\infty}\mu_{a,a',b}=\beta^{(1)}_{a-a',a'}$ *if* $a'<a-\frac{r-1}{2}$.

Proof For the proof, we will use the spherical Fourier transform.
1) For the first part, we need to show that, for $t\in\mathbb{R}^r$ such that $\sum_{i=1}^r t_i>1+k(r-k)-\frac{k(r+1)}{2}$, $1\le k\le r$,

$$\lim_{a\to 0}\frac{{}_3F_2(a',a'-t,b;a'+b,a+a';e)}{{}_3F_2(a',a',b;a'+b,a+a';e)}=0.$$

In fact, as ${}_3F_2(a',a'-t,b;a'+b,a+a';e)=\sum_{m\ge 0}\frac{(a'-t)_m(b)_m d_m}{(a'+b)_m(\frac{n}{r})_m}\times\frac{(a')_m}{(a+a')_m}$, then using the fact that $a\mapsto\frac{(a')_m}{(a+a')_m}$ is a decreasing function of a on $(\frac{r-1}{2},\infty)$ and that

$$\sum_{m\ge 0}\left|\frac{(a'-t)_m(b)_m d_m}{(a'+b)_m(\frac{n}{r})_m}\right|<\infty,$$

the monotone convergence theorem enables us to invert sum and limit to get

$$\lim_{a\to 0}{}_3F_2(a',a'-t,b;a'+b,a+a';e)={}_2F_1(a'-t,b;a'+b;e).$$

Similarly, we show that

$$\lim_{a\to 0}{}_3F_2(a',a',b;a'+b,a+a';e)=\infty.$$

Therefore

$$\lim_{a\to 0}\mu_{a,a',b}=\delta_0.$$

On the other hand, when $a\to\infty$, all the terms in the numerator and in the denominator go to zero except the one corresponding to $m=(0,...,0)$ which is equal to 1. Thus spherical Fourier transform tends to 1, which implies that

$$\lim_{a\to\infty}\mu_{a,a',b}=\delta_e.$$

2) We will use the result established in Theorem 3.1, that is if $X \sim \mu_{a',a,b}$ is independent of $W \sim \beta^{(2)}_{b,a}$, then $\pi^{-1}(e+\pi(X)(W))(e) \sim \mu_{a,a',b}$. Thus according to the point 1) established above, we have that

$$\text{if} \quad \lim_{a'\to 0} \mu_{a',a,b} = \delta_0, \quad \text{then} \quad \lim_{a'\to 0} \mu_{a,a',b} = \delta_e.$$

With the same reasoning, we see that

$$\lim_{a'\to\infty} \mu_{a,a',b} = \beta^{(1)}_{a,b}.$$

3) Similarly, except the term corresponding to $m = (0, ..., 0)$, which is equal to 1, all the other terms in the numerator and in the denominator go to zero when b tends to zero. Thus the spherical Fourier transform tends to 1, which implies that

$$\lim_{b\to 0} \mu_{a,a',b} = \delta_e.$$

When $b \to \infty$, we have, for $t \in \mathbb{R}^r$ such that $\sum_{i=1}^k t_i + k(a-a') > 1 + k(r-k) - k\frac{(r+1)}{2}, \ 1 \leq k \leq r$,

$$\lim_{b\to\infty} \ {}_3F_2(a', a'-t, b; a'+b, a+a'; e) = \lim_{b\to\infty} \sum_{m\geq 0} \frac{(a'-t)_m (a')_m d_m}{(a+a')_m (\frac{n}{r})_m} \times \frac{(b)_m}{(a'+b)_m}.$$

Here also, we can invert the sum and the limit to obtain that

$$\lim_{b\to\infty} \ {}_3F_2(a', a'-t, b; a'+b, a+a'; e) = \ {}_2F_1(a', a'-t; a+a'; e).$$

Similarly,

$$\lim_{b\to\infty} \ {}_3F_2(a', a', b; a'+b, a+a'; e) = \ {}_2F_1(a', a'; a+a'; e).$$

In the case where $a' < a - \frac{r-1}{2}$, the spherical Fourier transform of $\mu_{a,a',b}$ tends to

$$\frac{{}_2F_1(a', a'-t; a+a'; e)}{{}_2F_1(a', a'; a+a'; e)} = \frac{\Gamma_\Omega(a)\Gamma_\Omega(a-a'+t)}{\Gamma_\Omega(a-a')\Gamma_\Omega(a+t)},$$

which is the spherical Fourier transform of $\beta^{(1)}_{a-a',a'}$.
Otherwise the spherical Fourier transform of $\mu_{a,a',b}$ tends to 0, in which case

$$\lim_{b\to\infty} \mu_{a,a',b} = \delta_0.$$

□

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